

# On Building superpotentials in F- GUTs

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## Abstract

Using characters of finite group representations, we construct the fusion algebras of operators of the spectrum of F- theory GUTs. These fusion relations are used in building monodromy invariant superpotentials of the low energy effective 4d  $\mathcal{N} = 1$  supersymmetric GUT models.

*Key words: F-GUT models, characters of finite groups, fusion algebra of operators, superpotentials.*

## 1 Introduction

Ten dimensional superstring theory compactified to 4d space-time gives a basic framework to describe elementary particle interactions in four dimensions at  $M_{GUT}$  scale. In this framework, one can build phenomenologically viable supersymmetric Grand Unified Models (GUT) with *discrete symmetries* covering results on minimal supersymmetric standard model (MSSM) and aspects of neutrino physics [1] whose flavor mixing requires finite group symmetries like the alternating group  $A_4$  [2, 3, 4, 5]. One also disposes of dual ways for engineering GUT models involving more fundamental objects giving different, but equivalent, manners to approach the idea of super-unification; for example by using heterotic string vacua where both gravity and gauge dynamics descend from the closed string sector; or by using type II strings with gauge degrees of freedom localised on D-branes wrapping cycles of compact space. The string theory approach offers therefore new tools beyond quantum field theory method to deal with the usual difficulties in

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constructing GUT models [6]. With the new ingredients of 10d string theories compactified to 4d space-time; and depending on the strength of the string couplings  $g_s$ , various proposals have been developed to engineer stringy inspired 4d models extending MSSM; and where discrete symmetries have a geometric interpretation and are implemented in a natural way.

In the perturbative region of the string landscape where  $g_s$  is small, two particular approaches have been subject to intensive investigations; these are: (i) the approach based on  $E_8 \times E_8$  heterotic string taking advantage of the exceptional gauge symmetry to realise the idea of grand unified theory with GUT symmetries of type  $SU(5)$  Georgi-Glashow symmetry, flipped  $SU_5 \times U(1)$ ,  $SO(10)$  and  $E_6$ . These GUT symmetry candidates are all of them subgroups of one of the two  $E_8$ 's of heterotic string theory [7, 8]. (ii) the approach based on perturbative type IIA/B string orientifolds exploiting the localisation of the gauge degrees of freedom along the D-branes [9, 10, 11]. Though it accommodates MSSM gauge group in a nice manner, the second construction cannot implement exceptional gauge symmetry in terms of D-branes. However, the difficulty to relate the  $E_8$  of heterotic string to type IIB D-brane construction can be overcome by thinking of  $g_s$  as a dynamical coupling that varies over the compactification space. In this non perturbative regime, type IIB compactification with 7-branes is described by F-theory [12] where aspects of 7-branes get geometrised in terms of properties of elliptic Calabi-Yau fourfolds (CY4) with an  $E_8$  geometry. This link is because at strong string coupling, new degrees of freedom, namely the  $(p, q)$ -strings [13], become light and realise exceptional gauge symmetries even in a theory based on branes; a special feature that makes F-theory on CY4s with exceptional singularity a remarkable framework for the study of supersymmetric GUT models building.

In the last few years, it has been shown that the set of four-dimensional solutions of F-theory on elliptically Calabi-Yau fourfolds  $\mathcal{Y}_4$  with exceptional  $E_8$  geometry constitutes a particularly interesting class of string vacua for embedding supersymmetric GUTs in string theory. The basic properties of the exceptional elliptic model singles out F-theory GUT as the prototype where difficulties of intersecting brane models in perturbative orientifolds are overcome. More recently, there has been an important development in embedding GUT-models with some special discrete symmetries  $\Gamma$  including the alternating  $A_4$  group privileged for neutrino flavor mixing [2, 3]. These kinds of discrete groups  $\Gamma$  emerge naturally in F-theory compactification on elliptically Calabi-Yau 4-folds  $\mathcal{Y}_4$  with a threefold base  $\mathcal{B}_3$  and an  $E_8$  geometry [14]-[26]; they are captured by monodromy of 2-cycles in the compact sector of F-theory on CY4. In this GUT building, one considers an

exceptional 7-brane wrapping divisors of  $\mathcal{B}_3$  and focus on the effective 8-dimensional supersymmetric gauge theory on  $\mathbb{R}^{1,3} \times \mathcal{S}_{GUT}$  combined with tools borrowed from heterotic spectral covers construction [28, 29, 30]. The unified gauge theory  $G_{GUT} \times \Gamma$  lives on wrapped 7-brane on GUT surface  $\mathcal{S}_{GUT}$  with gauge symmetry  $G_{GUT}$  controlled by specific structure of the singularity over  $\mathcal{S}_{GUT}$ . This singularity is given by the discriminant of the elliptic fibration and is determined by the Kodaira's classification of singular fibers [31, 32]. The GUT gauge symmetry  $G_{GUT}$  is engineered by partial unfolding of the  $E_8$  singularity of the CY4 into  $G_{GUT} \times U(1)^n$  with maximal abelian  $U(1)^n \subset G_\perp$  and where  $G_\perp$  is the commutant of  $G_{GUT}$  in  $E_8$ . The discrete symmetry  $\Gamma$  is given by subgroups of the finite Weyl group of  $G_\perp$ . In the particular example  $G_{GUT} \times G_\perp = SO_{10} \times SU_4^\perp$ , matter curves of the GUT model are given by the decomposition of the **248** adjoint representation of  $E_8$  in terms of  $SO_{10} \times SU_4^\perp$  representations namely

$$\begin{aligned} \mathbf{248} \quad \rightarrow \quad & (\mathbf{45}, \mathbf{1}_\perp) \oplus (\mathbf{1}, \mathbf{15}_\perp) \oplus \\ & (\mathbf{16}, \mathbf{4}_\perp) \oplus (\overline{\mathbf{16}}, \overline{\mathbf{4}}_\perp) \oplus (\mathbf{10}, \mathbf{6}_\perp) \end{aligned} \quad (1.1)$$

A similar decomposition can be also written down for the GUT models with gauge symmetry  $G_{GUT} = SU_5$  and commutant  $G_\perp = SU_5^\perp$ . There, the **248** adjoint representation of  $E_8$  is broken down into  $SU_5 \times SU_5^\perp$  representations; and matter curves localised on brane intersections are associated with  $(\mathbf{10}, \mathbf{5}_\perp)$ ,  $(\overline{\mathbf{5}}, \mathbf{10}_\perp)$ ; their adjoints and  $(\mathbf{1}, \mathbf{24}_\perp)$ .

In addition to gauge group  $G_{GUT}$  representations, models of F-theory GUTs have a finite spectrum  $\{\Phi_{R_i}\}$  indexed by quantum numbers of monodromy group  $\Gamma$ . In the example of  $SO_{10} \times \Gamma$  models [27]-[41], the possible monodromies  $\Gamma$  are given by subgroups of the permutation symmetry  $\mathbb{S}_4$ ; and so have at most 5 irreducible representations  $\mathbf{R}_i$ . For  $SU_5 \times \Gamma'$  models, the  $\Gamma$ 's are sub-symmetries of  $\mathbb{S}_5$  having at most 7 irreducible representations. As shown on (1.1), some of these  $\Phi_{R_i}$ 's are somehow special in the sense they are scalars under the gauge symmetry but carry non trivial charges under  $\Gamma$ . These special fields representations, often called flavons, are also interesting in F-theory GUT; in particular in the study of neutrino physics and in the engineering of hierarchy [4, 5]. These flavons have been also interpreted as extra fields of the Higgs sector like in extended MSSM; and have been used for dealing with GUT constraints such as proton decay [42, 43]. By requiring invariance under  $\Gamma$ , one then disposes of an important tool to construct chiral superfields  $\Phi_{R_i}$  couplings including flavons; the  $\Gamma$ -invariance condition controls therefore the structure of the superpotentials  $W = W(\Phi_{R_i})$  of the supersymmetric GUT models since it will permit some interactions between  $\Phi_{R_i}$  and forbids others. However, to build monodromy invariant superpotentials  $W$  of the underlying low energy effective 4d  $\mathcal{N} = 1$  supersymmetric QFT, one has to perform tensor products  $\otimes_i \Phi_{R_i}$  of representations  $R_i$  of the monodromy group  $\Gamma$ ; and then takes

the trace. These computations require using fusion rules like

$$\Phi_{R_i} \otimes \Phi_{R_j} = \sum_{R_k} c_{R_i, R_j}^{R_k} \Phi_{R_k} \quad (1.2)$$

which, to our knowledge, have not been enough studied in F- GUT literature [44, 4]. It is then interesting to explore this area and determine these fusion rules and the corresponding closed algebras  $\mathcal{F}_\Gamma$  for those discrete groups  $\Gamma$  involved in F- theory compactifications on CY4s.

In this paper, we derive the closed fusion algebras  $\mathcal{F}_\Gamma$  of operators of the F- theory GUT spectrum  $\{\Phi_{R_i}\}$  by using algebraic properties of the  $\Phi_{R_i}$ 's; in particular the characters  $\chi_{R_i}$  of the group representations  $R_i$  of monodromy  $\Gamma$  and their dimensions. First, we show how these operator fusion algebras  $\mathcal{F}_\Gamma$  can be constructed; and as applications, we give the explicit list of the  $\mathcal{F}_\Gamma$ 's for those monodromy symmetries involved in the construction of superpotentials in F-GUT; in particular for the cases of non abelian finite groups like the symmetric groups  $\mathbb{S}_5, \mathbb{S}_4, \mathbb{S}_3$ ; the alternating  $\mathbb{A}_5, \mathbb{A}_4$ ; and the dihedral  $\mathbb{D}_4$ . We also give the fusion algebras  $\mathcal{F}_{\mathbb{Z}_N}$  associated with the particular abelian groups  $\mathbb{Z}_N$  as a matter to complete the study.

The presentation is as follows: In section 2, we give some useful tools and properties on models of F-theory GUTs. First we recall the main lines of F-theory on elliptic CY4s and the algebraic geometry approach using the Tate form of the elliptic fibration. Then we describe the idea of spectral covers construction in F-theory GUTs and show how it is used in practice for  $SU_5 \times \Gamma$  models and  $SO_{10} \times \Gamma'$  models. In section 3, we consider the example of  $\mathbb{S}_4$  permutation group and describe how this discrete symmetry appears as monodromy in F-GUT; and how it is involved in building superpotentials  $W(\Phi_{R_i})$ . Then, we use characters of the 5 irreducible  $\mathbb{S}_4$ - representations to build the underlying  $\mathcal{F}_{\mathbb{S}_4}$  algebra of merging operators. In section 4, we derive the  $\mathcal{F}_\Gamma$ 's for the non abelian  $\mathbb{A}_4, \mathbb{D}_4$  and  $\mathbb{S}_3$  appearing also in the engineering of F-GUTs. In section 5, we construct the  $\mathcal{F}_\Gamma$ 's for higher order groups; in particular  $\mathbb{S}_5$  and  $\mathbb{A}_5$ . In section 6, we conclude and make a comment on the fusion algebra for abelian monodromies like  $\mathbb{Z}_N$ . In section 7, we give an appendix where useful tools on discrete groups are collected.

## 2 General on F- theory GUTs

Since its discovery in 1996, F- theory [12] and its compactifications on elliptically fibered Calabi-Yau manifolds to lower space-time dimensions have been subject to huge interest

because of several stringy and geometric properties; in particular for their dualities with M-theory [45] and heterotic string

$$\text{F-theory/K3} \leftrightarrow \text{heterotic string}/T^2 \quad (2.1)$$

and also for those aspects linking brane physics with exceptional symmetry groups to the homology of Calabi-Yau fourfolds with 4-form  $G_4$  flux [46]. The twelve dimensional F-theory compactified on Calabi-Yau manifolds, which may be thought of as a non perturbative description of a class of string vacua, can be motivated in various manners; too particularly as a strongly coupled type IIB string theory with 7-branes and varying dilaton. By using string dualities, it may be also linked to M-theory on a vanishing 2-torus [47, 48, 49],

$$T^2 = S_A^1 \times S_B^1 \quad (2.2)$$

or remarkably to  $E_8 \times E_8$  heterotic string theory (2.1) where one disposes of basic results on engineering of vector bundles on elliptically fibered Calabi-Yau 3-folds via the spectral covers construction [28, 29, 30].

In F-theory compactification, it is conjectured that physics of type IIB orientifold [33, 29] on complex  $n$ -fold  $\mathcal{B}_n$  with 7-branes is encoded in the geometry of an  $n+1$ -fold  $\mathcal{Y}_{n+1}$  given by a complex elliptic curve  $\mathcal{E}$  fibered on the complex  $\mathcal{B}_n$  base

$$\begin{array}{ccc} \mathcal{E} & \rightarrow & \mathcal{Y}_{n+1} \\ & \downarrow & \\ & \mathcal{B}_n & \end{array} \quad (2.3)$$

The curve fiber  $\mathcal{E}$  is not part of the physical space-time; but a clever trick that accounts for the variation of the complex structure  $\tau$  of  $\mathcal{E} \sim T^2$  with the two following features: (i) the usual geometric  $SL(2, Z)$  action on the real 2-torus  $T^2$  is identified with the well known  $SL(2, Z)$  symmetry of 10d type IIB supergravity supporting S-duality property. (ii) the complex  $\tau$  is realised in terms of complex axio-dilaton field like

$$\tau = C + ie^{-\phi} \quad (2.4)$$

with axion  $C$ , dilaton  $\phi$  and type IIB string coupling constant  $e^\phi$ . In this geometric representation, the location of the 7-branes of IIB orientifold theory corresponds to singular value of the axio-dilaton  $\tau$ ; which, from cycle-homology view, corresponds as well to the shrinking of a 1-cycle of the elliptic fiber  $\mathcal{E}$ . Thus, the degeneration locus of the curve  $\mathcal{E}$  in F-theory on elliptically fibered Calabi-Yau manifolds  $\mathcal{Y}_{n+1}$  describes the presence of 7-branes wrapping cycles in the base  $\mathcal{B}_n$ . A simple example is given by F-theory on complex K3 surface modeling physics of type IIB orientifolds in  $8d$  space-time dimensions. Another interesting example corresponds to  $4d$ -space-time models given by F-theory on an elliptically fibered Calabi-Yau fourfolds  $\mathcal{Y}_4$  that captures the physics of type IIB orientifolds on complex 3-folds  $\mathcal{B}_3$ .

## 2.1 Tate models

F-theory compactified on CY4s is modelled by using Weierstrass curve whose useful features are nicely exhibited by using its Tate form. To fix idea, we think it interesting to first review briefly some useful aspects on F-theory and its compactification on elliptic CY4 hosting  $\mathcal{N} = 1$  supersymmetric GUTs; then turn to describe the main line lines of Tate models.

- *Weierstrass equation*

Generally speaking, one of powerful features of F-theory is that it combines two basic ingredients coming from two apparently different sources; one from type II string and the other from heterotic string; these are:

- a) localisation of degrees of freedom as described in perturbative models of type II orientifolds with D-branes,
- b) exceptional gauge groups and spectral covers construction as used in  $E_8 \times E_8$  heterotic string.

It happens that these two features are the essence of a mathematical theorem [34] which states that every elliptic fibration like (2.3) can be represented by a Weierstrass model with an underlying exceptional geometry. In the particular case of Calabi-Yau fourfolds  $\mathcal{Y}_4$  based on threefolds  $\mathcal{B}_3$ , the corresponding elliptic fibration with an  $E_8$  geometry is described by the Weierstrass equation

$$y^2 = x^3 + fxz^4 + gz^6 \quad (2.5)$$

with  $(y, x, z)$  homogeneous coordinates of the weighted projective space  $WP_{2,3,1}$ . The complex  $f$  and  $g$ , specifying the shape of the elliptic curve, are respectively holomorphic sections of  $\mathbb{H}^0(\mathcal{B}_3, \mathcal{K}^{-4})$  and  $\mathbb{H}^0(\mathcal{B}_3, \mathcal{K}^{-6})$  with  $\mathcal{K}$  the canonical bundle of the base  $\mathcal{B}_3$  [21]. In the case where the base  $\mathcal{B}_3$  is covered by some local complex coordinates  $\{u_i\}$ , the complex holomorphic sections  $f$  and  $g$  are given by suitable polynomials

$$f = f(u_i) \quad , \quad g = g(u_i) \quad (2.6)$$

and moreover the elliptic fibration (2.5) can be put into a Tate form where the underlying  $E_8$  gauge symmetry is broken down to some gauge group  $G_{GUT}$  along the complex surface divisor  $\mathcal{S}_{GUT}$  of the base space  $\mathcal{B}_3$  of the fibration; see eq(2.9) reported below. Notice also that in the coordinates patch where  $z = 1$ ; eq(2.5) reduces to

$$y^2 = x^3 + fx + g \quad (2.7)$$

with discriminant  $\Delta$  given by the usual formula

$$\Delta = 27g^2 + 4f^3 \quad (2.8)$$

The zero values of this discriminant  $\Delta$  play an important role in the F-GUT construction. By thinking of  $\Delta$  in terms of a product of factors like  $\prod_i \Delta_i$ , its zeros describe the loci  $\Delta_i = 0$  where the elliptic curve  $\mathcal{E}$  degenerates. These loci are interpreted in terms of locations of the 7-branes wrapping some divisors  $\mathcal{D}_i$  of the  $\mathcal{B}_3$ - base of the CY4. The extra non compact directions of the 7-branes fill the  $4d$  space- time where live GUT models. To get more insight into these brane/geometry features and on the way the gauge symmetry groups and matter localisations emerge in F-theory description, it is interesting to use the Tate form of the elliptic curve fiber  $\mathcal{E}$  that we describe in what follows.

- *Tate representation*

A convenient way to exhibit explicitly the singularities of the elliptic fibration (2.5) is to use the Tate form of the elliptic curve; it is given by the following complex holomorphic equation [35]

$$y^2 = x^3 + a_1xyz + a_2x^2z^2 + a_3yz^3 + a_4xz^4 + a_6z^6 \quad (2.9)$$

which is related to Weierstrass eq(2.5) by coordinates redefinition. Like for the complex  $f$  and  $g$ , the new complex holomorphic sections  $a_n = a_n(u_i)$  depend on the complex coordinates  $u_i$  of base  $\mathcal{B}_3$ ; they encode properties of the discriminant loci  $\Delta_i = 0$  of the above elliptic fibration obtained by solving the condition

$$\Delta = -\frac{1}{4}\beta_2^2(\beta_2\beta_6 - \beta_4^2) - 8\beta_4^3 - 27\beta_6^2 + 9\beta_2\beta_6\beta_4 \quad (2.10)$$

where we have set

$$\begin{aligned} \beta_2 &= a_1^2 + 4a_2 \\ \beta_4 &= a_1a_3 + 2a_4 \\ \beta_6 &= a_3^2 + 4a_6 \end{aligned} \quad (2.11)$$

Notice that  $f$  and  $g$  of (2.8) are related to the  $a_n$ 's like

$$\begin{aligned} f &= \frac{1}{24}(\beta_2^2 - 24\beta_4) \\ g &= -\frac{1}{864}(36\beta_2^2\beta_4 - \beta_3^2 - 216\beta_6) \end{aligned} \quad (2.12)$$

As noticed before the discriminant (2.10-2.11) can, under some assumptions<sup>1</sup> on the  $a_n$ 's, be factorized into product of factors like  $\prod_i \Delta_i$ ; each factor  $\Delta_i$  describing the location of

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<sup>1</sup>Near GUT surface  $\mathcal{S}_{GUT}$  of the  $SU_5$  model defined by the divisor  $w = 0$ , the  $a_n$  holomorphic sections of  $\mathcal{B}_3$  have the typical form  $a_n \sim w^{5-k} \mathbf{b}_{k,n}$  where the new  $\mathbf{b}_{k,n} = \mathbf{b}_{k,n}(u_1, u_2, w)$  are as in table (2.13).

a 7- brane on a divisor  $\mathcal{D}_i$  in the complex  $3d$  base  $\mathcal{B}_3$ ; one of them is the GUT surface  $\mathcal{S}_{GUT}$ . Two divisors  $\mathcal{D}_i, \mathcal{D}_j$  may intersect on curves  $\Sigma_{ij} = \mathcal{D}_i \cap \mathcal{D}_j$  where fundamental matter localise; while three divisors may intersect at points  $P_{ijk} = \mathcal{D}_i \cap \mathcal{D}_j \cap \mathcal{D}_k$  corresponding to Yukawa couplings. It turns out that the gauge symmetry group on GUT surface  $\mathcal{S}_{GUT}$  is precisely encoded by the vanishing degree of the  $a_n$  sections on  $\mathcal{S}_{GUT}$ . For the examples of  $SU(5)$  and  $SO(10)$  gauge symmetries along GUT divisor  $\mathcal{S}_{GUT}$  given by  $w = 0$ ; we have the following behaviors,; for more details see [35, 36, 37]

group	$a_1$	$a_2$	$a_3$	$a_4$	$a_6$	
$SU(5)$	$\mathbf{b}_5$	$\mathbf{b}_4 w$	$\mathbf{b}_3 w^2$	$\mathbf{b}_2 w^3$	$\mathbf{b}_0 w^5$	(2.13)
$SO(10)$	$\mathbf{b}_5 w$	$\mathbf{b}_4 w$	$\mathbf{b}_3 w^2$	$\mathbf{b}_2 w^3$	$\mathbf{b}_0 w^5$	

where the complex  $\mathbf{b}_k$ 's generically depend on all coordinates of  $\mathcal{B}_3$  but do not contain an overall factor of the complex variable  $w$ . Using the expression of the  $a_k$ 's in terms of  $\mathbf{b}_k$  and  $w$ , the discriminant  $\Delta_{su5}$  of the elliptic fibration with  $SU_5$  symmetry on GUT surface  $\mathcal{S}_{GUT}$  factorises as

$$\Delta_{su5} = -w^5 \times \delta \quad (2.14)$$

with  $w^5$  encoding  $SU_5$  symmetry and the factor  $\delta$  describing a single- component locus of an  $I_1$  singularity of Kodaira classification; it reads as follows

$$\delta = \mathbf{b}_5^4 \mathcal{P} + w \mathbf{b}_5^2 (8 \mathbf{b}_4 \mathcal{P} + \mathbf{b}_5 \mathcal{R}) + w^2 (16 \mathbf{b}_3^2 \mathbf{b}_4^2 + \mathbf{b}_5 \mathcal{Q}) + \mathcal{O}(w^3) \quad (2.15)$$

where

$$\begin{aligned} \mathcal{P} &= \mathbf{b}_3^2 \mathbf{b}_4 - \mathbf{b}_2 \mathbf{b}_3 \mathbf{b}_5 + \mathbf{b}_0 \mathbf{b}_5^2 \\ \mathcal{R} &= 4 \mathbf{b}_0 \mathbf{b}_4 \mathbf{b}_5 - \mathbf{b}_3^3 - \mathbf{b}_2^2 \mathbf{b}_5 \end{aligned} \quad (2.16)$$

From the factorisation (2.14), we learn that the cohomology class  $[\Delta]$  of the discriminant  $\Delta_{su5}$  in terms of the GUT divisor  $[S]$  and the  $I_1$  singularity divisor  $[S_1]$  is given by the sum

$$[\Delta] = 5[S] + [S_1] \quad (2.17)$$

## 2.2 Spectral covers in $SO_{10}$ and $SU_5$ models

In this subsection, we first describe the main lines of spectral covers in F- theory on Calabi-Yau fourfolds. Then, we focus on how the construction works on the example of GUT- models embedded in F-theory compactifications. We restrict this description to those  $G \times \Gamma$  models with discrete  $\Gamma$  and gauge invariance  $G = SU_5, SO_{10}$ .



### 2.2.1 Spectral covers in F- theory

In F- theory based models building, spectral covers approach provides a tricky manner to: (i) determine the various matter representations  $\mathcal{R}_i$  localised along the matter curves  $\Sigma_i$  as exhibited by eq(2.27) given below; and (ii) engineer the flux required by chirality feature of the model in terms of few parameters; see eqs(2.23-2.25). If one is interested only in the physics on the GUT surface  $\mathcal{S}_{GUT}$ , the key idea of the spectral covers construction proceeds as follows:

- first, use Tate form (2.9) with holomorphic sections  $a_n$  of table (2.13) to fix the desired gauge symmetry  $G$  on GUT surface  $\mathcal{S}_{GUT}$ . For the case where  $G = SU(5)$ , we have

$$y^2 = x^3 + b_5xy + b_4x^2w + b_3yw^2 + b_2xw^3 + b_0w^5 \quad (2.18)$$

In this case the initial  $E_8$  singularity of the elliptic fibration of the CY4 has been lifted to  $G = SU(5) \times U(1)^4$  with  $U(1)^4$  standing for the Cartan charges of the perpendicular  $SU(5)^\perp$ , the commutant of the  $SU(5)$  gauge symmetry inside  $E_8$ .

- second, restrict the Tate model to the neighbourhood of the divisor  $\mathcal{S}_{GUT} \subset \mathcal{B}_3$  by using spectral covers method. The latter is inspired from spectral covers construction used in building models embedded in heterotic string theory [29]. In F-theory, the idea of the spectral covers method relies on zooming into the local neighbourhood of the  $w = 0$  divisor  $\mathcal{S}_{GUT}$  inside  $\mathcal{B}_3$  by dropping out all terms of higher power in the normal coordinate  $w$  that appear in the sections  $\mathbf{b}_n$ ; that is restricting them to

$$b_n = \mathbf{b}_n|_{w=0} \quad (2.19)$$

where now  $b_n$  live on  $\mathcal{S}_{GUT}$ .

- then, think of  $\mathcal{S}_{GUT}$  as the base of the bundle  $\mathcal{K}_S \rightarrow \mathcal{S}_{GUT}$ , with GUT surface given by  $s = 0$ ; and approach the neighbourhood of  $\mathcal{S}_{GUT}$  in terms of spectral surfaces  $\mathcal{C}_n$  given by divisors of the total space. In case of  $SU_5$  model, the integer  $n$  takes the values 5, 10, 20 respectively associated with 10-plets  $10_{t_i}$ , 5-plets  $5_{t_i+t_j}$  and charged flavons  $\vartheta_{t_i-t_j}$ . For the example of  $\mathcal{C}_5$ , describing the fundamental representation of  $SU(5)^\perp$ , we have

$$\mathcal{C}_5 = b_0s^5 + b_1s^4 + b_2s^3 + b_3s^2 + b_4s + b_5 = 0 \quad (2.20)$$

with  $b_1 = 0$  required by traceless property of  $SU(5)$ ; a feature that is implemented explicitly by factorising  $\mathcal{C}_5$  as follows

$$\mathcal{C}_5 = b_0 \prod_{i=1}^5 (s - t_i) \quad (2.21)$$

and requiring

$$t_1 + t_2 + t_3 + t_4 + t_5 = 0. \quad (2.22)$$

One can think about above  $\mathcal{C}_5$ , whose expression (2.21) is manifestly invariant under  $\mathbb{S}_5$  permuting the 5 roots  $t_i$ , as encoding information about the discriminant locus in the local vicinity of GUT surface.

- finally, fluxes are engineered by splitting spectral covers  $\mathcal{C}_n$  like  $\prod_k \mathcal{C}_{n_k}$  with  $n = \sum_k n_k$  and where each factor  $\mathcal{C}_{n_k}$  has an expansion in terms of the spectral variable  $s$  as in eq(2.20). This splitting introduce new holomorphic sections obeying constraints following by equating the expansion of  $\mathcal{C}_n$  with the one resulting from  $\prod_k \mathcal{C}_{n_k}$ . As an example, the splittings of  $\mathcal{C}_5$  and corresponding monodromy groups are given by

splitted spectral covers	monodromy
$\mathcal{C}_4^{(5)} \times \mathcal{C}_1^{(5)}$	$\mathbb{S}_4$
$\mathcal{C}_3^{(5)} \times \mathcal{C}_2^{(5)}$	$\mathbb{S}_3 \times \mathbb{S}_2$
$\mathcal{C}_3^{(5)} \times \mathcal{C}_1^{(5)} \times \mathcal{C}_1^{(5)}$	$\mathbb{S}_3$
$\mathcal{C}_2^{(5)} \times \mathcal{C}_2^{(5)} \times \mathcal{C}_1^{(5)}$	$\mathbb{S}_2 \times \mathbb{S}_2$
$\mathcal{C}_2^{(5)} \times \mathcal{C}_1^{(5)} \times \mathcal{C}_1^{(5)} \times \mathcal{C}_1^{(5)}$	$\mathbb{S}_2$
$\mathcal{C}_1^{(5)} \times \mathcal{C}_1^{(5)} \times \mathcal{C}_1^{(5)} \times \mathcal{C}_1^{(5)} \times \mathcal{C}_1^{(5)}$	-

(2.23)

where for instance

$$\begin{aligned} \mathcal{C}_4^{(5)} &= \alpha_0 s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4 \\ \mathcal{C}_1^{(5)} &= \beta_0 s + \beta_1 \end{aligned} \quad (2.24)$$

and similar relations for the others. The  $\alpha_l$ 's and  $\beta_l$ 's are new holomorphic sections; they are related to the  $b_l$ 's like

$$\begin{aligned} b_0 &= \alpha_0 \beta_0 & , & & b_1 &= \alpha_0 \beta_1 + \alpha_1 \beta_0 \\ b_2 &= \alpha_1 \beta_1 + \alpha_2 \beta_0 & , & & b_3 &= \alpha_2 \beta_1 + \alpha_3 \beta_0 \\ b_4 &= \alpha_3 \beta_1 + \alpha_4 \beta_0 & , & & b_5 &= \alpha_4 \beta_1 \end{aligned} \quad (2.25)$$

### *Localised matter*

In the spectral covers description introduced just above, the various matter representations  $\mathcal{R}_n$  localised along the matter curves  $\Sigma_n$  are determined by the intersections  $\mathcal{C}_n \cap \mathcal{S}_{GUT}$ . For the example of the fundamental  $\mathcal{C}_5$ , the intersection with the GUT surface  $\mathcal{S}_{GUT}$  is given by the relation

$$b_5 = -b_0 t_1 t_2 t_3 t_4 t_5 = 0 \quad (2.26)$$

having five solutions given by  $t_i = 0$ . These solutions describe precisely the localisation of five tenplet matter curves of the  $SU(5) \times \mathbb{S}_5$  model

$$10_{t_1}, 10_{t_2}, 10_{t_3}, 10_{t_4}, 10_{t_5} \quad (2.27)$$

By extending this spectral covers construction to the other representations of  $SU(5)^\perp$  involved in the breaking of  $E_8$  down to  $SU(5) \times SU(5)^\perp$ , in particular to the antisymmetric and adjoint ones, one can also describe in quite similar manner the localisation of the other matter multiplets namely the  $(\bar{\mathbf{5}}, \mathbf{10}_\perp)$  and  $(\mathbf{1}, \mathbf{24}_\perp)$ ; for explicit details see [38] and refs therein.

#### *heterotic dual*

First notice that not any F-theory compactification has a heterotic string dual. For those cases of F-theory compactifications having heterotic string duals; the elliptic Calabi-Yau fourfolds  $\mathcal{Y}_4 : \mathcal{E} \rightarrow \mathcal{B}_3$  have also a K3-fibration over a complex surface  $\mathcal{B}_2$  as follows

$$\mathcal{Y}_4 : K3 \rightarrow \mathcal{B}_2 \quad (2.28)$$

By thinking of the complex surface  $K3$  in term of the elliptic fibration of a real 2-torus  $T^2$  over a real 2-sphere  $S^2$ ; or more precisely in terms of a complex elliptic curve over projective line like  $K3 : \mathcal{E} \rightarrow \mathbb{P}^1$ ; it follows that the complex 3d base space is in turns given by a fibration of a complex projective line  $\mathbb{P}^1$  on complex base surface  $\mathcal{B}_2$  as follows

$$\mathcal{B}_3 : \mathbb{P}^1 \rightarrow \mathcal{B}_2 \quad (2.29)$$

This fibration of  $\mathcal{B}_3$  puts therefore a strong restriction on the set of Calabi-Yau fourfolds of F-theory GUT models having heterotic string duals.

#### *ALE fibration*

The fibration (2.29) is very suggestive in dealing with local models of F-theory-GUT. There, one has a quite similar local structure of  $\mathcal{B}_3$  near  $\mathcal{S}_{GUT}$  since the role of  $\mathcal{B}_2$  is done but  $\mathcal{S}_{GUT}$ ; and the role of  $\mathbb{P}^1$  in (2.29) gets now played by several intersecting  $\mathbb{P}_i^1$ 's glued as in the graph of Dynkin diagram of Lie algebra  $E_8$ . In the limit where all sizes of the  $\mathbb{P}_i^1$ 's are shrunk to zero, one is left with an  $E_8$  singularity of the elliptic fibration.

$$\mathcal{B}_3 : ALE|_{E_8} \rightarrow \mathcal{S}_{GUT} \quad (2.30)$$

By blowing up the size of some of the  $\mathbb{P}_i^1$ 's, one can engineer desired gauge symmetries  $G$  given by subgroups of  $E_8$ . Therefore, the complex surface  $\mathcal{S}_{GUT}$  can be locally viewed as the basis of an ALE fibration which describes the singularity structure along  $\mathcal{S}_{GUT}$ . The

ALE fiber contains a distinguished set of two-cycles  $\gamma_i$  with intersection  $\gamma_i \circ \gamma_j$  given by minus the Cartan matrix of  $E_8$ .

$$\gamma_i \circ \gamma_j = -K_{ij}(E_8) \quad (2.31)$$

If a number  $r$  of two-cycles  $\gamma'_i$  among the eight ones have non-zero size; the  $E_8$  symmetry is broken to subgroups  $G_{8-r} \times U(1)^r$ ; for example where  $r = 4$ , the  $E_8$  symmetry breaks down to  $SU_5 \times U(1)^4$ ; and for  $r = 3$  it breaks to  $SO_{10} \times U(1)^3$ . Moreover, matter curves and Yukawa coupling points on the divisor  $\mathcal{S}_{GUT}$  exhibit enhanced gauge symmetries; they correspond to brane- intersections where localise matter and Yukawa interactions. In what follows, we consider the cases of  $G \times \Gamma$  models with  $G = SO_{10}, SU_5$ ; and comment briefly on the spectral cover construction for some discrete monodromies  $\Gamma$ .

### 2.2.2 $G \times \Gamma$ models: $G = SO_{10}, SU_5$

Focusing first on the family  $SO_{10} \times \Gamma$  models of F-theory GUTs with  $SO_{10}$  gauge symmetry, the candidates for discrete monodromy  $\Gamma$  is given by one of the 30 possible subgroups of the symmetric group  $\mathbb{S}_4$ ; the Weyl group group of  $SU_4^\perp$ . To make an idea on the explicit list of these monodromy  $\Gamma$ 's, see eq(2.42) reported below and also the  $\mathbb{S}_4$ - branch in fig.1 giving particular subgroups of  $\mathbb{S}_5$ .

- $SO_{10} \times \Gamma$  models

The matter content  $\{\Phi_{R_i}\}$  of the  $SO_{10} \times \Gamma$  models are read from the decomposition (1.1); it is labeled by four weights  $t_i$  like

$$\Phi_{R_i} \quad : \quad \mathbf{16}_{t_i}, \quad \mathbf{16}_{-t_i}, \quad \mathbf{10}_{t_i+t_j}, \quad \mathbf{1}_{t_i-t_j} \quad (2.32)$$

with traceless condition

$$t_1 + t_2 + t_3 + t_4 = 0 \quad (2.33)$$

The components of the four sixteen-plets  $\mathbf{16}_{t_i} \equiv \{\mathbf{16}_{t_1}, \mathbf{16}_{t_2}, \mathbf{16}_{t_3}, \mathbf{16}_{t_4}\}$  and those of the six ten-plets  $\mathbf{10}_{t_i+t_j} \equiv \{\mathbf{10}_{\pm(t_1+t_2)}, \mathbf{10}_{\pm(t_1+t_3)}, \mathbf{10}_{\pm(t_2+t_3)}\}$  as well as the 15 singlets (flavons) are related to each other by monodromies  $\Gamma$ . These discrete symmetries offer a framework of approaching  $SO_{10} \times \Gamma$  models embedded in F-theory compactified on elliptic Calabi-Yau fourfolds

$$CY4 \sim E_{SO_{10}} \times \mathcal{B}_3 \quad (2.34)$$

with complex 3- dim base  $\mathcal{B}_3$  containing  $\mathcal{S}_{GUT}$ . The Tate form of this Calabi-Yau fourfolds is realised as follows

$$y^2 = x^3 + b_5xyw + b_4x^2w + b_3yw^2 + b_2xw^3 + b_0w^5 \quad (2.35)$$

with holomorphic sections  $b_k$  living on the GUT surface. The homology classes of  $x$ ,  $y$ ,  $w$  and  $b_k$  are expressed in terms of the Chern class  $c_1 = c_1(\mathcal{S}_{GUT})$  of the tangent bundle of the  $\mathcal{S}_{GUT}$  surface; and the Chern class  $-t$  of the normal bundle  $\mathcal{N}_{\mathcal{S}_{GUT}|\mathcal{B}_3}$  as follows

$$\begin{aligned} [y] &= 3(c_1 - t) \\ [x] &= 2(c_1 - t) \\ [w] &= -t \\ [b_k] &= (6c_1 - t) - kc_1 \end{aligned} \tag{2.36}$$

Matter curves in  $SO_{10} \times \Gamma$  models are described by spectral covers of GUT surfaces. To each of the multiplets in (2.32); it is associated a spectral cover  $\mathcal{C}_n$  given by an order  $n$  holomorphic polynomial in a spectral variable  $s$ ; with number of roots given by dimension of corresponding  $SU_4^\perp$  representation. For example, the spectral cover  $\mathcal{C}_4$  associated with the four sixteen-plets  $\mathbf{16}_{t_i}$  is given by

$$\mathcal{C}_4 : b_0 s^4 + b_1 s^3 + b_2 s^2 + b_3 s + b_4 = 0 \tag{2.37}$$

with  $b_1 = 0$  due to traceless condition of  $SU_4^\perp$ . This polynomial factorises like

$$\mathcal{C}_4 = b_0 \prod_{i=1}^4 (s - t_i) \tag{2.38}$$

where the  $t_i$  zeros are precisely as in eq(2.33). The last expression of  $\mathcal{C}_4$  is manifestly invariant under  $S_4$  permuting the 4 roots  $t_i$ . Matter curves  $\mathbf{16}_{t_i}$  are given by the intersection of  $\mathcal{C}_4$  with the GUT surface  $\mathcal{S}_{GUT}$  realised in this formulation by the divisor  $s = 0$ ; that is

$$\mathcal{C}_4 \cap \mathcal{S}_{GUT} \quad \Rightarrow \quad b_4 = 0 \quad \Rightarrow \quad b_0 \prod_{i=1}^4 t_i = 0 \tag{2.39}$$

Similar expression can we written down for the other spectral covers; for the example of the six tenplets  $\mathbf{10}_{t_i+t_j}$ ; the corresponding spectral cover is given by

$$\mathcal{C}_6 = \sum_{k=1}^6 d_k s^{6-k} \tag{2.40}$$

with the six matter curves  $\mathbf{10}_{t_i+t_j}$  on GUT surface localised at the zeros of  $\mathcal{C}_6$  as shown below

$$\mathcal{C}_6 = d_0 \prod_{i < j=1}^4 (s - t_i - t_j) \tag{2.41}$$

Notice that the breaking of monodromy induced by non trivial fluxes is engineered by splitting spectral cover method as in eq(2.23) regarding the spectral cover of  $SU_5^\perp$ . In

the case of fundamental  $\mathcal{C}_4$  of  $SU_4^\perp$ ; we have quite similar decompositions; for examples  $\mathcal{C}_4 = \mathcal{C}_3^{(4)} \times \mathcal{C}_1^{(4)}$  reducing  $\mathbb{S}_4$  monodromy to  $\mathbb{S}_3$ ; and  $\mathcal{C}_4 = \mathcal{C}_2^{(4)} \times \mathcal{C}_2^{(4)}$  reducing  $\mathbb{S}_4$  monodromy to  $\mathbb{S}_2 \times \mathbb{S}_2$ .

•  $SU_5 \times \Gamma$  models

An analogous description of  $SO_{10} \times \Gamma$  models can be done for other F-theory GUTs. In the interesting case of the  $SU_5 \times \Gamma$  models the gauge symmetry is given by Georgi-Glashow group  $SU_5$ , and monodromy groups  $\Gamma$  contained in  $\mathbb{S}_5$ . A particular branch of subgroups  $\Gamma$  is the one contained in  $\mathbb{S}_4$  listed in the following table, see fig. 1

$\Gamma$	order	multiplicity
$\mathbb{S}_4$	24	1
$\mathbb{A}_4$	12	1
$\mathbb{D}_4$	8	3
$\mathbb{S}_3$	6	4
$\mathbb{V}_4$	4	1
$\mathbb{Z}_4$	4	3
$\mathbb{Z}_2 \times \mathbb{Z}_2$	4	3
$\mathbb{Z}_3$	3	4
$\mathbb{Z}_2$	2	9
$I_{id}$	1	1

(2.42)

The elliptically fibered Calabi-Yau fourfolds  $CY4 \sim E_{SU_5} \times \mathcal{B}_3$  is locally realised by the Tate model; it is described by the algebraic equation  $y^2 = x^3 + b'_5 xy + b'_4 x^2 w + b'_3 y w^2 + b'_2 x w^3 + b'_0 w^5$  with sections  $b'_n$  given by holomorphic functions on  $\mathcal{B}_3$  with properties as in eqs(2.14)-(2.17).

The matter curves of these models are read from the decomposition of the **248** adjoint representation of  $E_8$  in terms of  $SU_5 \times SU_5^\perp$  representations as given below

$$\begin{aligned} \mathbf{248} \rightarrow & (\mathbf{24}, \mathbf{1}_\perp) \oplus (\mathbf{1}, \mathbf{24}_\perp) \oplus \\ & (\mathbf{10}, \mathbf{5}_\perp) \oplus (\bar{\mathbf{5}}, \mathbf{10}_\perp) \oplus (\bar{\mathbf{10}}, \bar{\mathbf{5}}_\perp) \oplus (\mathbf{5}, \bar{\mathbf{10}}_\perp) \end{aligned} \quad (2.43)$$

In this  $SU_5$  theory, the monodromy symmetry  $\Gamma$  is contained in the Weyl group of the perpendicular  $SU_5^\perp$ ; and the matter content of the model is labeled by five weights  $t_i$  like

$$\mathbf{10}_{t_i}, \quad \bar{\mathbf{10}}_{-t_i}, \quad \bar{\mathbf{5}}_{t_i+t_j}, \quad \mathbf{5}_{-t_i-t_j}, \quad \mathbf{1}_{t_i-t_j} \quad (2.44)$$

with traceless condition

$$t_1 + t_2 + t_3 + t_4 + t_5 = 0 \quad (2.45)$$

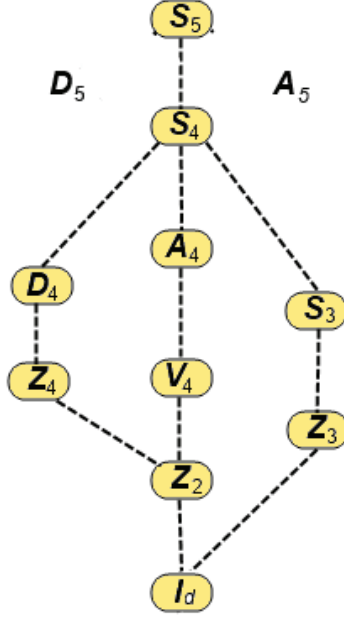


Figure 1: A branch of the tree of the 156 subgroups of the symmetric  $S_5$ . The branch with top vertex  $S_4$ , gives the 30 subgroups of the symmetric  $S_4 \subset S_5$ .

The spectral cover description of the matter curves (2.44) is quite similar to the above  $SO_{10} \times \Gamma$  models one; for the fundamental spectral cover  $\mathcal{C}_5$  of  $SU_5 \times \mathbb{S}_5$  models, see eqs(2.20-2.21).

With these tools at hand, we are now in position to construct the fusion algebras  $\mathcal{F}_\Gamma$  of operators of the F- theory GUT spectrum  $\{\Phi_{R_i}\}$  by mainly focussing on  $SO_{10} \times \Gamma$  models by starting with largest  $\Gamma = \mathbb{S}_4$ ; quite similar constructions are valid for  $SU_5 \times \Gamma$  theory with  $\Gamma \subset \mathbb{S}_5$ .

### 3 Fusion operators algebra with $\mathbb{S}_4$ symmetry

To begin, notice that we have chosen to start by studying the fusion algebra  $\mathcal{F}_{\mathbb{S}_4}$  of the set  $\{\Phi_{R_i}\}$  carrying quantum numbers in  $\mathbb{S}_4$ . Though the corresponding  $SO_{10} \times \mathbb{S}_4$  model is not phenomenologically interesting since only three matter generations are known and so  $\mathbb{S}_4$  should be broken down; we take the opportunity to illustrate how the  $\Phi_{R_i}$ 's are involved in building superpotentials; and to derive the corresponding fusion algebra  $\mathcal{F}_{\mathbb{S}_4}$ . To that purpose, we first give useful tools on  $\mathbb{S}_4$  representations as involved in F-GUT; and turn after to build the  $\mathbb{S}_4$ - fusion algebra  $\mathcal{F}_{\mathbb{S}_4}$ .

### 3.1 $\mathbb{S}_4$ as F-GUT monodromy

The permutation symmetry  $\mathbb{S}_4$  is a discrete group having  $24$  elements arranged into  $5$  conjugacy classes  $\mathfrak{C}_1, \dots, \mathfrak{C}_5$  as shown on table (7.1) reported in appendix. It has  $5$  irreducible representations  $\mathbf{R}_1, \dots, \mathbf{R}_5$  with dimensions  $d_i$  given by the character relation linking the order of  $\mathbb{S}_4$  to the  $\mathbf{R}_i$  dimensions like  $24 = \sum_i d_i^2$ ; by expanding we have

$$24 = 1^2 + (1')^2 + 2^2 + 3^2 + (3')^2 \quad (3.1)$$

The  $\mathbb{S}_4$  group has  $3$  non commuting generators  $a, b, c$  which can be chosen as respectively given by the 2- cycle (12), the 3-cycle (123) and the 4- cycle (1234). These generators obey amongst others the cyclic relations  $a^2 = b^3 = c^4 = I_{id}$ ; their characters  $\chi_{\mathbf{R}_i}(a)$ ,  $\chi_{\mathbf{R}_i}(b)$ ,  $\chi_{\mathbf{R}_i}(c)$  can be read from eq(7.1); they are given by

$\chi_{\mathbf{R}_i}^\alpha$	$\chi_I$	$\chi_3$	$\chi_2$	$\chi_{3'}$	$\chi_\epsilon$
$a$	1	1	0	-1	-1
$b$	1	0	-1	0	1
$c$	1	-1	0	1	-1

(3.2)

The  $\chi_{\mathbf{R}_i}^\alpha$  can be organised into  $5$  character vectors  $\tilde{\chi}_i = (\chi_i^a, \chi_i^b, \chi_i^c)$  where we have set  $\chi_i^\alpha = \chi_{\mathbf{R}_i}(\alpha)$ . Observe also the following remarkable property that turns out to play an important role in the derivation of the fusion algebra for  $\mathbb{S}_4$  monodromy,

$$\tilde{\chi}_3 + \tilde{\chi}_{3'} = (0, 0, 0) \quad (3.3)$$

These  $\tilde{\chi}_i$ 's will be used in this paper as a tool to characterise the curves spectrum of  $SO_{10} \times \mathbb{S}_4$  model. Indeed, by following [27, 38], see also eqs(2.32-2.33), the matter curves in the spectrum of this supersymmetric model involve three kinds of multiplets<sup>2</sup> transforming in representations of  $\mathbb{S}_4$ ; these are:

- the  $4$  matter curves generally denoted as  $\mathbf{16}_\mu$ ; they describe the 16-plets of the  $SO_{10} \times \mathbb{S}_4$  model; three of them interpreted in terms of the usual GUT generations;
- the  $6$  Higgs curves  $\mathbf{10}_{[\mu\nu]}$  describing the 10-plets; the corresponding low energy (super) fields can have VEVs giving masses to particles of GUT; and
- the  $3+12$  curves  $\mathbf{1}_{ij}$  describing the flavons, denoted by  $4 \times 4$  traceless matrix  $\vartheta_\mu^\nu$ ; three of them neutral, and the  $12 = 6 + 6'$ , denoted below as  $\tilde{\vartheta}_\mu^\nu$ , are charged under  $\mathbb{S}_4$ ; they can have VEVs; they are important for generating mass hierarchy.

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<sup>2</sup>These multiplets are interpreted in the underlying low energy effective 4d  $\mathcal{N} = 1$  QFT in terms of chiral superfields  $\Phi_{R_i}$  carrying gauge quantum numbers; but also monodromy representations.



The curves spectrum of  $SO_{10} \times \mathbb{S}_4$  model is commonly presented as on following table

matters curves	weights	homology	$U(1)_X$ flux
$16_\mu$	$t_\mu$	$\eta - 4c_1$	0
$10_{[\mu\nu]}$	$(t_\mu + t_\nu) _{\mu \leq \nu}$	$\eta' - 6c_1$	0
$\tilde{\vartheta}_\mu^\nu$	$t_\mu - t_\nu$	$\eta'' - 12c_1$	0

(3.4)

where we have also given the homology classes (2.36); whose interpretations can be found in [17, 27, 38]; the last column is trivial here; but it is important for the study of  $SO_{10} \times \Gamma$  models with monodromy  $\Gamma$  given by subgroups of  $\mathbb{S}_4$ . There, the  $U(1)_X$  flux takes non zero values and splits the spectral covers.

### 3.2 Superpotentials and fusion algebra $\mathcal{F}_{\mathbb{S}_4}$

To study the building of superpotentials  $W(\Phi_{R_i})$  of low energy effective 4d  $\mathcal{N} = 1$  supersymmetric theory of  $SO_{10} \times \mathbb{S}_4$  model, it is interesting to re-construct the structure of table (3.4). By using the irreducible  $\mathbf{R}_i$  representations of  $\mathbb{S}_4$  as well as their  $\chi_i^\alpha$  characters (3.2), we can reformulate the curve spectrum of  $SO_{10} \times \mathbb{S}_4$ - model as follows

matters curves	$S_4$ irreps	character $\vec{\chi}_i$	homology	$U(1)_X$ flux
$16_0$	$1^+$	$(1, 1, 1)$	$-c_1$	0
$16_i$	$3^+$	$(1, 0, -1)$	$\eta - 3c_1$	0
$10_i$	$3^+$	$(1, 0, -1)$	$\eta' - 3c_1$	0
$10_{[ij]}$	$3^-$	$(-1, 0, 1)$	$-3c_1$	0
$1_i$	$3^+$	$(1, 0, -1)$	$\eta'' - 3c_1$	0
$1_{[ij]}$	$3^-$	$(-1, 0, 1)$	$-3c_1$	0

(3.5)

where we have also used the reduction  $\mathbf{12} = 2 \times (\mathbf{3}^+ \oplus \mathbf{3}^-)$ ; see below for its derivation. We will also use the convenient notations suggested by  $\mathbb{S}_4$ - characters (3.4),

$$\begin{aligned}
\mathbf{1} &\equiv \mathbf{1}_{(1,1,1)}^+ & , & \quad \mathbf{3} &\equiv \mathbf{3}_{(1,0,-1)}^+ & , & \quad \mathbf{2} &\equiv \mathbf{2}_{(0,-1,0)}^0 \\
\boldsymbol{\epsilon} &\equiv \mathbf{1}_{(-1,1,-1)}^- & , & \quad \mathbf{3}' &\equiv \mathbf{3}_{(-1,0,1)}^- & , & 
\end{aligned}
\tag{3.6}$$

- *Superpotentials*  $W(\Phi_{R_i})$

Superpotentials  $W(\Phi_{R_i})$  of the low energy effective QFT of  $SO_{10} \times \mathbb{S}_4$ - models are given by product of matter chiral superfield operators  $\Psi_{\mathbf{16}_\mu^M} \sim \mathbf{16}_\mu^M$ , Higgs  $\Psi_{\mathbf{10}_{[\mu\nu]}^H} \sim \mathbf{10}_{[\mu\nu]}^H$  and flavons  $\Psi_{\mathbf{1}_{\alpha\beta}^F} \sim \mathbf{1}_{\alpha\beta}^F \equiv \boldsymbol{\vartheta}_{\alpha\beta}$ . These superpotentials should be  $SO_{10}$  gauge invariant; but also invariant under monodromy  $\mathbb{S}_4$ . A typical example of a gauge invariant superpotential in  $SO_{10} \times \mathbb{S}_4$ - model is given by the tree level (top-quark) Yukawa couplings

$$W_{SO_{10}}^{tree} = \lambda^{\mu\nu\rho\sigma} \mathbf{16}_\mu^M \otimes \mathbf{16}_\nu^M \otimes \mathbf{10}_{[\mu\nu]}^H \tag{3.7}$$

where  $\lambda^{\mu\nu\rho\sigma}$  are coupling constants transforming as a rank 4 tensor; invariance under  $\mathbb{S}_4$  puts constraint on these  $\lambda^{\mu\nu\rho\sigma}$ 's. Other forms of superpotentials can be also written down; they involve flavons like in the following non renormalisable 4-order one

$$W_{SO_{10}}^{(4)} = \lambda^{\mu\nu\rho\sigma\alpha\beta} \mathbf{16}_\mu^M \otimes \mathbf{16}_\nu^M \otimes \mathbf{10}_{[\mu\nu]}^H \otimes \boldsymbol{\vartheta}_{[\alpha\beta]}^F \quad (3.8)$$

In what follows, we develop a method to construct  $\mathbb{S}_4$ -monodromy invariant superpotentials; this approach is based on the fusion algebra  $\mathcal{F}_{\mathbb{S}_4}$ ; itself based on the characters of irreducible representations of  $\mathbb{S}_4$ . To derive  $\mathcal{F}_{\mathbb{S}_4}$ , we proceed as follows:

- (i) we start from the finite spectrum (3.5); and denote the corresponding superfield operators by their representations and characters like

matters curves	$SO_{10}$	$\mathbb{S}_4$ irreps	$SO_{10} \times \mathbb{S}_4$
$16_{(1,1,1)}$	16	$1^+$	$16 \otimes 1_{(1,1,1)}^+$
$16_{(1,0,-1)}$		$3^+$	$16 \otimes 3_{(1,0,-1)}^+$
$10_{(1,0,-1)}$	10	$3^+$	$10 \otimes 3_{(1,0,-1)}^+$
$10_{(-1,0,1)}$		$3^-$	$10 \otimes 3_{(-1,0,1)}^-$
$\boldsymbol{\vartheta}_{(1,0,-1)}$	1	$3^+$	$1 \otimes 3_{(1,0,-1)}^+$
$\boldsymbol{\vartheta}_{(-1,0,1)}$		$3^-$	$1 \otimes 3_{(-1,0,1)}^-$

(3.9)

this means that the operator  $\mathbf{16}_{(1,1,1)}$  is a trivial singlet of  $\mathbb{S}_4$ ; the  $\mathbf{16}_{(1,0,-1)}$  is a  $\mathbf{3}^+$ -triplet of  $\mathbb{S}_4$ ; the  $\mathbf{10}_{(-1,0,1)}$  is a  $\mathbf{3}^-$ -triplet of  $\mathbb{S}_4$ ; and so on.

- (ii) seen that  $\mathbf{16}$ ,  $\mathbf{10}$  and  $\boldsymbol{\vartheta}$  are representations of  $SO_{10}$ ; and seen that we are interested in invariance under discrete  $\mathbb{S}_4$ , we will refer to these superfield operators like

$$\mathbf{16}_{(p_1,q_1,r_1)} \quad , \quad \mathbf{10}_{(p_2,q_2,r_2)} \quad , \quad \boldsymbol{\vartheta}_{(p_3,q_3,r_3)} \quad \rightarrow \quad \boldsymbol{\Phi}_{(p_i,q_i,r_i)} \quad (3.10)$$

In other words, think of  $\mathbf{16}_{(p_1,q_1,r_1)}$  as given by  $\mathbf{16} \otimes \boldsymbol{\Phi}_{(p_1,q_1,r_1)}$ ; and so on. Then focus on the algebra of the  $\mathbb{S}_4$  representations; the properties of the  $SO_{10}$  gauge multiplets are dealt as in usual GUT models.

- (iii) Generic superpotentials  $W(\Phi_{R_i})$  in  $\mathbb{S}_4$ -model are given by

$$W_{SO_{10}} = \sum tr [\mathcal{R}_{(P,Q,R)}] \quad (3.11)$$

with

$$\mathcal{R}_{(P,Q,R)} = \boldsymbol{\Phi}_{(p_1,q_1,r_1)} \otimes \cdots \otimes \boldsymbol{\Phi}_{(p_n,q_n,r_n)} \quad (3.12)$$

where the trace refers both to invariance under gauge symmetry; and  $\mathbb{S}_4$ -monodromy. An example of  $\mathcal{R}_{(P,Q,R)}$  is given by the Yukawa tri-coupling

$$W_{SO_{10}}^{tree} = \mathbf{16}_{(p_1,q_1,r_1)}^M \otimes \mathbf{16}_{(p_2,q_2,r_2)}^M \otimes \mathbf{10}_{(p_3,q_3,r_3)}^H \quad (3.13)$$

If we take all these  $\mathcal{I}$  operators as  $\mathbb{S}_4$ - triplets like

$$\begin{aligned} \mathbf{16}_{(p_1, q_1, r_1)}^M &= \mathbf{16}_{(1, 0, -1)} \\ \mathbf{16}_{(p_2, q_2, r_2)}^M &= \mathbf{16}_{(1, 0, -1)} \\ \mathbf{10}_{(p_3, q_3, r_3)}^H &= \mathbf{10}_{(-1, 0, 1)} \end{aligned} \quad (3.14)$$

we end with the following reducible 27-dim representation of  $\mathbb{S}_4$

$$\Phi_{(1, 0, -1)} \otimes \Phi'_{(1, 0, -1)} \otimes \Phi''_{(-1, 0, 1)} = \sum_{(p, q, r)} n_{(p, q, r)} \mathbf{R}_{(p, q, r)} \quad (3.15)$$

where the positive integers  $n_{(p, q, r)}$  are some multiplicities  $n_{(p, q, r)}$  constrained by the total dimension of the tensor product of representations. Other examples of chiral superpotentials are given by higher order superpotentials involving flavons; for instance  $W(\Phi_{R_i}) = W_{SO_{10}}^{tree} + W_{SO_{10}}^{(4)}$  given by the sum (3.7) + (3.8).

• *Fusion algebra  $\mathcal{F}_{\mathbb{S}_4}$*

To compute the explicit expression of (3.11), one needs reducing the tensor product  $\mathcal{R}_{(P, Q, R)}$  in terms of a direct sum over the 5 irreducible representations of  $\mathbb{S}_4$  as follows

$$\mathcal{R}_{(P, Q, R)} = n_e \mathbf{1}_{(1, 1, 1)}^+ \oplus n_\epsilon \mathbf{1}_{(-1, 1, -1)}^- \oplus n_2 \mathbf{2}_{(0, -1, 0)}^0 \oplus n_+ \mathbf{3}_{(1, 0, -1)}^+ \oplus n_- \mathbf{3}_{(-1, 0, 1)}^- \quad (3.16)$$

The  $n_i$ 's are obtained by demanding two conservation laws; total dimension and total character. But to fully achieve the reduction of (3.11), one must know the fusion algebra of two operators  $\Phi_{(p_i, q_i, r_i)} \otimes \Phi_{(p_j, q_j, r_j)}$ ; then proceed step by step until getting the full reduction as above. In other words, it is enough to know the right hand of the following expansion

$$\Phi_{(p_i, q_i, r_i)} \otimes \Phi_{(p_j, q_j, r_j)} = \sum_k C_{\{(p_i, q_i, r_i)\}, \{(p_j, q_j, r_j)\}}^{\{(p_k, q_k, r_k)\}} \Phi_{(p_k, q_k, r_k)} \quad (3.17)$$

By using the irreducible  $\mathbf{R}_i$  representations of  $\mathbb{S}_4$ , the above fusion equation reads in a condensed manner as follows

$$\Phi_{\mathbf{R}_i} \otimes \Phi_{\mathbf{R}_j} = \sum_{\mathbf{R}_k} C_{\mathbf{R}_i, \mathbf{R}_j}^{\mathbf{R}_k} \Phi_{\mathbf{R}_k} \quad (3.18)$$

or formally like

$$\mathbf{R}_i \otimes \mathbf{R}_j = \oplus_{\mathbf{R}_k} C_{\mathbf{R}_i, \mathbf{R}_j}^{\mathbf{R}_k} \mathbf{R}_k \quad (3.19)$$

As one of the results of this paper is that the fusion algebras  $\mathcal{F}_{\mathbb{S}_4}$  of the  $\mathbb{S}_4$  monodromy symmetry is given, in addition to  $\mathbf{1}_{(1, 1, 1)}^+ \otimes \mathbf{R}_{(p, q, r)} = \mathbf{R}_{(p, q, r)}$ , by the following relations

preserving dimensions and characters

$$\begin{aligned}
\mathbf{3}^\pm &\otimes \mathbf{3}^\pm = \mathbf{1}^+ \oplus \mathbf{2}^0 \oplus \mathbf{3}^+ \oplus \mathbf{3}^- \\
\mathbf{3}^\pm &\otimes \mathbf{2}^0 = \mathbf{3}^+ \oplus \mathbf{3}^- \\
\mathbf{3}^\pm &\otimes \mathbf{1}^- = \mathbf{3}^\mp \\
\mathbf{3}^+ &\otimes \mathbf{3}^- = \mathbf{1}^- \oplus \mathbf{2}^0 \oplus \mathbf{3}^+ \oplus \mathbf{3}^- \\
\mathbf{2}^0 &\otimes \mathbf{2}^0 = \mathbf{1}^+ \oplus \mathbf{1}^- \oplus \mathbf{2}^0 \\
\mathbf{2}^0 &\otimes \mathbf{1}^- = \mathbf{2}^0 \\
\mathbf{1}^- &\otimes \mathbf{1}^- = \mathbf{1}^+
\end{aligned} \tag{3.20}$$

By using (3.6), these relations read into a condensed manner like

$$\begin{aligned}
\mathbf{3}_{(1,0,-1)}^+ &\otimes \mathbf{3}_{(1,0,-1)}^+ = \mathbf{9}_{(1,0,1)}^+ \\
\mathbf{3}_{(1,0,-1)}^+ &\otimes \mathbf{3}_{(-1,0,1)}^- = \mathbf{9}_{(-1,0,-1)}^- \\
\mathbf{3}_{(1,0,-1)}^+ &\otimes \mathbf{2}_{(0,-1,0)}^0 = \mathbf{6}_{(0,0,0)}^0 \\
\mathbf{3}_{(1,0,-1)}^+ &\otimes \mathbf{1}_{(-1,1,-1)}^- = \mathbf{3}_{(-1,0,1)}^- \\
\mathbf{3}_{(-1,0,1)}^- &\otimes \mathbf{3}_{(-1,0,1)}^- = \mathbf{9}_{(1,0,1)}^+ \\
\mathbf{3}_{(-1,0,1)}^- &\otimes \mathbf{2}_{(0,-1,0)}^0 = \mathbf{6}_{(0,0,0)}^0 \\
\mathbf{2}_{(0,-1,0)}^0 &\otimes \mathbf{2}_{(0,-1,0)}^0 = \mathbf{4}_{(0,1,0)}^0 \\
\mathbf{2}_{(0,-1,0)}^0 &\otimes \mathbf{1}_{(-1,1,-1)}^- = \mathbf{2}_{(0,-1,0)}^0 \\
\mathbf{1}_{(-1,1,-1)}^- &\otimes \mathbf{1}_{(-1,1,-1)}^- = \mathbf{1}_{(1,1,1)}^+
\end{aligned} \tag{3.21}$$

with right hand side given by the following

$$\begin{aligned}
\mathbf{9}_{(1,0,1)}^+ &= \mathbf{6}_{(0,0,0)}^0 \oplus \mathbf{2}_{(0,-1,0)}^0 \oplus \mathbf{1}_{(1,1,1)}^+ \\
\mathbf{9}_{(-1,0,-1)}^- &= \mathbf{6}_{(0,0,0)}^0 \oplus \mathbf{2}_{(0,-1,0)}^0 \oplus \mathbf{1}_{(-1,1,-1)}^- \\
\mathbf{6}_{(0,0,0)}^0 &= \mathbf{3}_{(1,0,-1)}^+ \oplus \mathbf{3}_{(-1,0,1)}^- \\
\mathbf{4}_{(0,1,0)}^0 &= \mathbf{2}_{(0,-1,0)}^0 \oplus \mathbf{1}_{(-1,1,-1)}^- \oplus \mathbf{1}_{(1,1,1)}^+
\end{aligned} \tag{3.22}$$

By substituting  $\mathbf{6}_{(0,0,0)}^0 = \mathbf{3}_{(1,0,-1)}^+ \oplus \mathbf{3}_{(-1,0,1)}^-$  back into  $\mathbf{9}_{(1,0,1)}^\pm$ ; we can read the  $n_{(p,q,r)}$  multiplicities of the irreducible representations (3.6) of  $\mathbb{S}_4$  monodromy

$$\begin{aligned}
\mathbf{9}_{(1,0,1)}^+ &= \mathbf{3}_{(1,0,-1)}^+ \oplus \mathbf{3}_{(-1,0,1)}^- \oplus \mathbf{2}_{(0,-1,0)}^0 \oplus \mathbf{1}_{(1,1,1)}^+ \\
\mathbf{9}_{(-1,0,-1)}^- &= \mathbf{3}_{(1,0,-1)}^+ \oplus \mathbf{3}_{(-1,0,1)}^- \oplus \mathbf{2}_{(0,-1,0)}^0 \oplus \mathbf{1}_{(-1,1,-1)}^-
\end{aligned} \tag{3.23}$$

From the operators fusion algebra (3.20), we learn that  $\mathbf{9}_{(1,0,1)}^+$  and  $\mathbf{4}_{(0,1,0)}^0$  have  $\mathbb{S}_4$ - monodromy invariants  $\mathbf{1}_{(1,1,1)}^+$ ; while  $\mathbf{9}_{(-1,0,-1)}^-$  and  $\mathbf{6}_{(0,0,0)}^0$  haven't. The explicit derivation of  $\mathcal{F}_{\mathbb{S}_4}$  is straightforward; it relies on requiring both sides of (3.20) to have same representation character and same dimension; these properties have been explicitly exhibited on (3.21-3.22).

## 4 Fusion algebras $\mathcal{F}_{\mathbb{A}_4}$ , $\mathcal{F}_{\mathbb{D}_4}$ , $\mathcal{F}_{\mathbb{S}_3}$

In this section, we extend the construction of (3.20-3.22) to non abelian subgroups of  $\mathbb{S}_4$ ; first we consider the alternating subgroup  $\mathbb{A}_4$ ; then  $\mathbb{S}_3$  and after the dihedral  $\mathbb{D}_4$ .

### 4.1 Algebra with $\mathbb{A}_4$ monodromy

The group  $\mathbb{A}_4$  is the  $\mathbb{S}_4$ - subgroup of even permutations; it has 12 elements arranged into 4 conjugacy classes  $\mathfrak{C}_1, \dots, \mathfrak{C}_4$  as on table (7.3) of appendix; it has 4 irreducible representations  $\mathbf{R}_1, \dots, \mathbf{R}_4$  with dimensions as in

$$12 = 1^2 + (1')^2 + (1'')^2 + 3^2 \quad (4.1)$$

Non abelian  $\mathbb{A}_4$  has 2 generators  $\alpha$  and  $\beta$  with characters like

$\mathbb{A}_4$	$\chi_1$	$\chi_{1'}$	$\chi_{1''}$	$\chi_3$
$\alpha$	1	1	1	-1
$\beta$	1	$j$	$j^2$	0

(4.2)

with  $j^3 = 1$ . By denoting the irreducible representations of  $\mathbb{A}_4$  as

$$\begin{aligned} \mathbf{1} &\equiv \mathbf{1}_{(1,1)}^0, & \mathbf{3} &\equiv \mathbf{3}_{(-1,0)}^0 \\ \mathbf{1}' &\equiv \mathbf{1}_{(1,j)}^+, & \mathbf{1}'' &\equiv \mathbf{1}_{(1,j^2)}^- \end{aligned} \quad (4.3)$$

we find that the fusion algebra  $\mathcal{F}_{\mathbb{A}_4}$  preserving dimensions and characters is given by

$$\begin{aligned} \mathbf{3}_{(-1,0)}^0 &\otimes \mathbf{3}_{(-1,0)}^0 = \mathbf{9}_{(1,0)}^0 \\ \mathbf{3}_{(-1,0)}^0 &\otimes \mathbf{1}_{(1,j^q)}^q = \mathbf{3}_{(-1,0)}^0 \\ \mathbf{1}_{(1,j^q)}^q &\otimes \mathbf{1}_{(1,j^p)}^p = \mathbf{1}_{(1,j^{p+q})}^{q+p} \end{aligned} \quad (4.4)$$

with

$$\mathbf{9}_{(1,0)}^0 = \mathbf{1}_{(1,1)}^0 \oplus \mathbf{1}_{(1,j)}^+ \oplus \mathbf{1}_{(1,j^2)}^- \oplus \mathbf{3}_{(-1,0)}^0 \oplus \mathbf{3}_{(-1,0)}^0 \quad (4.5)$$

where the three singlets appear once; and the triplet twice.

### 4.2 Fusion algebra $\mathcal{F}_{\mathbb{S}_3}$

The order 6 group  $\mathbb{S}_3$  has 3 conjugacy classes  $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3$ ; and 3 irreducible representations  $\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3$  as reported in (7.2) with dimensions read from the following relation

$$6 = 1^2 + (1')^2 + 2^2 \quad (4.6)$$

This finite group has two non commuting generators **a** and **b** with characters as follows

$\chi_{\mathbf{R}_i}^g$	$\chi_{\mathbf{I}}$	$\chi_{\mathbf{2}}$	$\chi_{\epsilon}$
$a$	1	0	-1
$b$	1	-1	1

(4.7)

Denoting the three representations like

$$\mathbf{1} \equiv \mathbf{1}_{(1,1)}^+ \quad , \quad \mathbf{1}' \equiv \mathbf{1}_{(-1,1)}^- \quad , \quad \mathbf{2} \equiv \mathbf{2}_{(0,-1)}^0 \quad (4.8)$$

we find that the fusion algebra  $\mathcal{F}_{\mathbb{S}_3}$  preserving dimension and character is given by

$$\begin{aligned} \mathbf{2}_{(0,-1)}^0 &\otimes \mathbf{2}_{(0,-1)}^0 = \mathbf{4}_{(0,1)}^0 \\ \mathbf{2}_{(0,-1)}^0 &\otimes \mathbf{1}_{(-1,1)}^- = \mathbf{2}_{(0,-1)}^- \\ \mathbf{1}_{(-1,1)}^- &\otimes \mathbf{1}_{(-1,1)}^- = \mathbf{1}_{(1,1)}^+ \end{aligned} \quad (4.9)$$

with

$$\mathbf{4}_{(0,1)}^0 = \mathbf{1}_{(1,1)}^+ \oplus \mathbf{1}_{(-1,1)}^- \oplus \mathbf{2}_{(0,-1)}^0 \quad (4.10)$$

Notice that  $\mathcal{F}_{\mathbb{S}_3}$  is a subalgebra of  $\mathcal{F}_{\mathbb{S}_4}$ ; this can be seen by comparing (4.9) with three last rows of (3.20).

### 4.3 Dihedral $\mathbb{D}_4$ symmetry

The  $\mathbb{D}_4$  is an order 8 subgroup of  $\mathbb{S}_4$ ; it has 5 irreducible representations as on

$$8 = (1_1)^2 + (1_2)^2 + (1_3)^2 + (1_4)^2 + 2^2 \quad (4.11)$$

it has 2 non commuting generators  $a, c$ , satisfying  $a^2 = 1$ ,  $c^4 = 1$ , and  $aca^{-1} = c^{-1}$ ; and 5 conjugation classes

$$\begin{aligned} \mathfrak{C}_1 &\equiv \{e\} & , & \quad \mathfrak{C}_2 \equiv \{c^2\} & , & \quad \mathfrak{C}_3 \equiv \{c, c^3\} \\ \mathfrak{C}_4 &\equiv \{a, c^2a\} & , & \quad \mathfrak{C}_5 \equiv \{ca, c^3a\} \end{aligned} \quad (4.12)$$

with character table as; see also appendix eq(7.4),

$\chi_{\mathbf{R}_i}^g$	$\chi_{\mathbf{1}_1}$	$\chi_{\mathbf{1}_2}$	$\chi_{\mathbf{1}_3}$	$\chi_{\mathbf{1}_4}$	$\chi_{\mathbf{2}}$
$a$	1	-1	1	-1	0
$c$	1	1	-1	-1	0

(4.13)

Denoting the 5 irreducible representations like

$$\begin{aligned} \mathbf{1}_1 &\equiv \mathbf{1}_{(1,1)} & , & \quad \mathbf{1}_3 \equiv \mathbf{1}_{(1,-1)} & , & \quad \mathbf{2} \equiv \mathbf{2}_{(0,0)} \\ \mathbf{1}_2 &\equiv \mathbf{1}_{(-1,1)} & , & \quad \mathbf{1}_4 \equiv \mathbf{1}_{(-1,-1)} & , & \end{aligned} \quad (4.14)$$

and solving the conditions for  $\mathbb{D}_4$ - fusion algebra  $\mathcal{F}_{\mathbb{D}_4}$  preserving dimension and character; we find

$$\begin{aligned} \mathbf{2}_{(0,0)} &\otimes \mathbf{2}_{(0,0)} = \mathbf{4}_{(0,0)} \\ \mathbf{2}_{(0,0)} &\otimes \mathbf{1}_{(p,q)} = \mathbf{2}_{(0,0)} \\ \mathbf{1}_{(p,q)} &\otimes \mathbf{1}_{(p',q')} = \mathbf{1}_{(pp',qq')} \end{aligned} \tag{4.15}$$

with the four following solutions

$$\begin{aligned} \mathcal{F}_{\mathbb{D}_4}^{(I)} &: \quad \mathbf{4}_{(0,0)} = \mathbf{1}_{(1,1)} \oplus \mathbf{1}_{(-1,-1)} \oplus \mathbf{1}_{(1,-1)} \oplus \mathbf{1}_{(-1,1)} \\ \mathcal{F}_{\mathbb{D}_4}^{(II)} &: \quad \mathbf{4}_{(0,0)} = \mathbf{1}_{(1,1)} \oplus \mathbf{1}_{(-1,-1)} \oplus \mathbf{2}_{(0,0)} \\ \mathcal{F}_{\mathbb{D}_4}^{(III)} &: \quad \mathbf{4}_{(0,0)} = \mathbf{1}_{(1,-1)} \oplus \mathbf{1}_{(-1,1)} \oplus \mathbf{2}_{(0,0)} \\ \mathcal{F}_{\mathbb{D}_4}^{(IV)} &: \quad \mathbf{4}_{(0,0)} = \mathbf{2}_{(0,0)} \oplus \mathbf{2}_{(0,0)} \end{aligned} \tag{4.16}$$

These relations teach us that generally speaking there are four fusion algebras  $\mathcal{F}_{\mathbb{D}_4}$ .

## 5 Extension to higher monodromies

In  $SU_5 \times \Gamma$  models; monodromies are contained in  $\mathbb{S}_5$ ; here we give two extensions of  $\mathcal{F}_{\mathbb{S}_4}$ ; we first give  $\mathcal{F}_{\mathbb{S}_5}$ , and then  $\mathcal{F}_{\mathbb{A}_5}$ .

### 5.1 Fusion algebra $\mathcal{F}_{\mathbb{S}_5}$

The group  $\mathbb{S}_5$  has 120 elements arranged into 7 conjugacy classes  $\mathfrak{C}_i$  as on (7.5); 7 irreducible representations  $\mathbf{R}_i$  with dimensions as in the expansion

$$120 = 1^2 + (1')^2 + 4^2 + (4')^2 + 5^2 + (5')^2 + 6^2 \tag{5.1}$$

It has 4 non commuting generators  $a, b, c, d$  which can be chosen as  $(12), (123), (1234), (12345)$ ; they obey amongst others the cyclic  $a^2 = b^3 = c^4 = d^2 = I_{id}$ ; their characters are as follows

$\chi_{\mathbf{R}_i}^g$	$\chi_1$	$\chi_{1'}$	$\chi_4$	$\chi_{4'}$	$\chi_5$	$\chi_{5'}$	$\chi_6$
$a$	1	-1	2	-2	1	-1	0
$b$	1	1	1	1	-1	-1	0
$c$	1	-1	0	0	-1	1	0
$d$	1	1	-1	-1	0	0	1

(5.2)

We denote the 7 irreducible representations like

$$\begin{aligned}
\mathbf{1} &= \mathbf{1}_{(1,1,1,1)}^+ & , & & \mathbf{1}' &= \mathbf{1}_{(-1,1,-1,1)}^- & , & & \mathbf{6} &= \mathbf{6}_{(0,0,0,1)}^0 \\
\mathbf{4} &= \mathbf{4}_{(2,1,0,-1)}^+ & , & & \mathbf{4}' &= \mathbf{4}_{(-2,1,0,-1)}^- & & & & \\
\mathbf{5} &= \mathbf{5}_{(1,-1,-1,0)}^+ & , & & \mathbf{5}' &= \mathbf{5}_{(-1,-1,1,0)}^- & & & & 
\end{aligned} \tag{5.3}$$

The fusion algebra  $\mathcal{F}_{\mathbb{S}_5}$  preserving dimensions and characters is big but closed; it reads in a condensed manner as follows:

$$\begin{aligned}
\mathbf{6}_{(0,0,0,1)}^0 &\otimes \mathbf{6}_{(0,0,0,1)}^0 &= \mathbf{36}_{(0,0,0,1)}^0 \\
\mathbf{6}_{(0,0,0,1)}^0 &\otimes \mathbf{5}_{(1,-1,-1,0)}^+ &= \mathbf{30}_{(0,0,0,0)}^0 \\
\mathbf{6}_{(0,0,0,1)}^0 &\otimes \mathbf{5}_{(-1,-1,1,0)}^- &= \mathbf{30}_{(0,0,0,0)}^0 \\
\mathbf{6}_{(0,0,0,1)}^0 &\otimes \mathbf{4}_{(2,1,0,-1)}^+ &= \mathbf{24}_{(0,0,0,-1)}^0 \\
\mathbf{6}_{(0,0,0,1)}^0 &\otimes \mathbf{4}_{(-2,1,0,-1)}^- &= \mathbf{24}_{(0,0,0,-1)}^0 \\
\mathbf{6}_{(0,0,0,1)}^0 &\otimes \mathbf{1}_{(-1,1,-1,1)}^- &= \mathbf{6}_{(0,0,0,1)}^0
\end{aligned} \tag{5.4}$$

and

$$\begin{aligned}
\mathbf{5}_{(1,-1,-1,0)}^+ &\otimes \mathbf{5}_{(1,-1,-1,0)}^+ &= \mathbf{25}_{(1,1,1,0)}^+ \\
\mathbf{5}_{(1,-1,-1,0)}^+ &\otimes \mathbf{5}_{(-1,-1,1,0)}^- &= \mathbf{25}_{(-1,1,-1,0)}^- \\
\mathbf{5}_{(1,-1,-1,0)}^+ &\otimes \mathbf{4}_{(2,1,0,-1)}^+ &= \mathbf{20}_{(2,-1,0,0)}^+ \\
\mathbf{5}_{(1,-1,-1,0)}^+ &\otimes \mathbf{4}_{(-2,1,0,-1)}^- &= \mathbf{20}_{(-2,-1,0,0)}^- \\
\mathbf{5}_{(1,-1,-1,0)}^+ &\otimes \mathbf{1}_{(-1,1,-1,1)}^- &= \mathbf{5}_{(-1,-1,1,0)}^- \\
\mathbf{5}_{(-1,-1,1,0)}^- &\otimes \mathbf{5}_{(-1,-1,1,0)}^- &= \mathbf{25}_{(1,1,1,0)}^+ \\
\mathbf{5}_{(-1,-1,1,0)}^- &\otimes \mathbf{4}_{(2,1,0,-1)}^+ &= \mathbf{20}_{(-2,-1,0,0)}^- \\
\mathbf{5}_{(-1,-1,1,0)}^- &\otimes \mathbf{4}_{(-2,1,0,-1)}^- &= \mathbf{20}_{(2,-1,0,0)}^+ \\
\mathbf{5}_{(-1,-1,1,0)}^- &\otimes \mathbf{1}_{(-1,1,-1,1)}^- &= \mathbf{5}_{(1,-1,-1,0)}^+
\end{aligned} \tag{5.5}$$

as well as

$$\begin{aligned}
\mathbf{4}_{(2,1,0,-1)}^+ &\otimes \mathbf{4}_{(2,1,0,-1)}^+ &= \mathbf{16}_{(4,1,0,1)}^+ \\
\mathbf{4}_{(2,1,0,-1)}^+ &\otimes \mathbf{4}_{(-2,1,0,-1)}^- &= \mathbf{16}_{(-4,1,0,1)}^- \\
\mathbf{4}_{(2,1,0,-1)}^+ &\otimes \mathbf{1}_{(-1,1,-1,1)}^- &= \mathbf{4}_{(-2,1,0,-1)}^- \\
\mathbf{4}_{(-2,1,0,-1)}^- &\otimes \mathbf{4}_{(-2,1,0,-1)}^- &= \mathbf{16}_{(4,1,0,1)}^+ \\
\mathbf{4}_{(-2,1,0,-1)}^- &\otimes \mathbf{1}_{(-1,1,-1,1)}^- &= \mathbf{4}_{(2,1,0,-1)}^+ \\
\mathbf{1}_{(-1,1,-1,1)}^- &\otimes \mathbf{1}_{(-1,1,-1,1)}^- &= \mathbf{1}_{(1,1,1,1)}^+
\end{aligned} \tag{5.6}$$



with right sides obtained by requiring conservation of characters; we find:

$$\begin{aligned}
\mathbf{36}_{(0,0,0,1)}^0 &= \mathbf{30}_{(0,0,0,0)}^0 \oplus \mathbf{6}_{(0,0,0,1)}^0 \\
\mathbf{30}_{(0,0,0,0)}^0 &= \mathbf{24}_{(0,0,0,-1)}^0 \oplus \mathbf{6}_{(0,0,0,1)}^0 \\
\mathbf{25}_{(1,1,1,0)}^+ &= \mathbf{24}_{(0,0,0,-1)}^0 \oplus \mathbf{1}_{(1,1,1,1)}^+ \\
\mathbf{25}_{(-1,1,-1,0)}^- &= \mathbf{24}_{(0,0,0,-1)}^0 \oplus \mathbf{1}_{(-1,1,-1,1)}^- \\
\mathbf{24}_{(0,0,0,-1)}^0 &= \mathbf{18}_{(0,0,0,-2)}^0 \oplus \mathbf{6}_{(0,0,0,1)}^0 \\
\mathbf{20}_{(2,-1,0,0)}^+ &= \mathbf{14}_{(2,-1,0,-1)}^+ \oplus \mathbf{6}_{(0,0,0,1)}^0 \\
\mathbf{20}_{(-2,-1,0,0)}^- &= \mathbf{14}_{(-2,-1,0,-1)}^- \oplus \mathbf{6}_{(0,0,0,1)}^0 \\
\mathbf{16}_{(4,1,0,1)}^+ &= \mathbf{10}_{(4,1,0,0)}^+ \oplus \mathbf{6}_{(0,0,0,1)}^0 \\
\mathbf{16}_{(-4,1,0,1)}^- &= \mathbf{10}_{(-4,1,0,0)}^- \oplus \mathbf{6}_{(0,0,0,1)}^0
\end{aligned} \tag{5.7}$$

where we have set

$$\begin{aligned}
\mathbf{18}_{(0,0,0,-2)}^0 &= \mathbf{10}_{(0,-2,0,0)}^0 \oplus \mathbf{4}_{(2,1,0,-1)}^+ \oplus \mathbf{4}_{(-2,1,0,-1)}^- \\
\mathbf{14}_{(2,-1,0,-1)}^+ &= \mathbf{10}_{(0,-2,0,0)}^0 \oplus \mathbf{4}_{(2,1,0,-1)}^+ \\
\mathbf{14}_{(-2,-1,0,-1)}^- &= \mathbf{10}_{(0,-2,0,0)}^0 \oplus \mathbf{4}_{(-2,1,0,-1)}^-
\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}
\mathbf{10}_{(0,-2,0,0)}^0 &= \mathbf{5}_{(1,-1,-1,0)}^+ \oplus \mathbf{5}_{(-1,-1,1,0)}^- \\
\mathbf{10}_{(4,1,0,0)}^+ &= \mathbf{5}_{(1,-1,-1,0)}^+ \oplus \mathbf{4}_{(2,1,0,-1)}^+ \oplus \mathbf{1}_{(1,1,1,1)}^+ \\
\mathbf{10}_{(-4,1,0,0)}^- &= \mathbf{5}_{(-1,-1,1,0)}^- \oplus \mathbf{4}_{(-2,1,0,-1)}^- \oplus \mathbf{1}_{(-1,1,-1,1)}^-
\end{aligned} \tag{5.9}$$

Putting eqs(5.7-5.9) back into eqs(5.4-5.6), one obtains the full explicit expression of the fusion algebra  $\mathcal{F}_{\mathbb{S}_5}$ .

## 5.2 Fusion algebra $\mathcal{F}_{\mathbb{A}_5}$

The alternating  $\mathbb{A}_5$  is an order 60 subgroup of the symmetric  $\mathbb{S}_5$ ; it has 5 conjugacy classes  $\mathfrak{C}_i$  and 5 irreducible representations  $\mathbf{R}_i$  with dimensions as in the expansion

$$60 = 1^2 + 3^2 + (3')^2 + 4^2 + 5^2 \tag{5.10}$$

The  $\mathbb{A}_5$  group has 3 non commuting generators  $\alpha, \beta, \gamma$ , obeying  $\alpha^2 = \beta^3 = \gamma^5 = \alpha\beta\gamma = 1$ , with characters given by real numbers as follows; see also table (7.6) in appendix,

$\chi_{\mathbf{R}_i}^g$	$\chi_1$	$\chi_3$	$\chi_{3'}$	$\chi_4$	$\chi_5$
$\alpha$	1	-1	-1	0	1
$\beta$	1	0	0	1	-1
$\gamma$	1	$\kappa_+$	$\kappa_-$	-1	0

(5.11)

where  $\kappa_{\pm} = \frac{1 \pm \sqrt{5}}{2}$ . We denote the 5 irreducible representations of  $\mathbb{A}_5$  like

$$\begin{aligned} \mathbf{1} &= \mathbf{1}_{(1,1,1)}^0, & \mathbf{3} &= \mathbf{3}_{(-1,0,\kappa_+)}^+, & \mathbf{3} &= \mathbf{3}_{(-1,0,\kappa_-)}^- \\ \mathbf{4} &= \mathbf{4}_{(0,1,-1)}^0, & \mathbf{5} &= \mathbf{5}_{(1,-1,0)}^0 \end{aligned} \quad (5.12)$$

The obtained fusion algebra  $\mathcal{F}_{\mathbb{A}_5}$  of the  $\mathbb{A}_5$ - irreducible representations is given by

$$\begin{aligned} \mathbf{5}_{(1,-1,0)}^0 &\otimes \mathbf{5}_{(1,-1,0)}^0 &= \mathbf{25}_{(1,1,0)}^0 \\ \mathbf{5}_{(1,-1,0)}^0 &\otimes \mathbf{4}_{(0,1,-1)}^0 &= \mathbf{20}_{(0,-1,0)}^0 \\ \mathbf{5}_{(1,-1,0)}^0 &\otimes \mathbf{3}_{(-1,0,\kappa_+)}^+ &= \mathbf{15}_{(-1,0,0)}^0 \\ \mathbf{5}_{(1,-1,0)}^0 &\otimes \mathbf{3}_{(-1,0,\kappa_-)}^- &= \mathbf{15}_{(-1,0,0)}^0 \end{aligned} \quad (5.13)$$

and

$$\begin{aligned} \mathbf{4}_{(0,1,-1)}^0 &\otimes \mathbf{4}_{(0,1,-1)}^0 &= \mathbf{16}_{(0,1,-1)}^0 \\ \mathbf{4}_{(0,1,-1)}^0 &\otimes \mathbf{3}_{(-1,0,\kappa_+)}^+ &= \mathbf{12}_{(0,0,-\kappa_+)}^+ \\ \mathbf{4}_{(0,1,-1)}^0 &\otimes \mathbf{3}_{(-1,0,\kappa_-)}^- &= \mathbf{12}_{(0,0,-\kappa_-)}^- \end{aligned} \quad (5.14)$$

as well as

$$\begin{aligned} \mathbf{3}_{(-1,0,\kappa_+)}^+ &\otimes \mathbf{3}_{(-1,0,\kappa_+)}^+ &= \mathbf{9}_{(1,0,\kappa_+^2)}^+ \\ \mathbf{3}_{(-1,0,\kappa_+)}^+ &\otimes \mathbf{3}_{(-1,0,\kappa_-)}^- &= \mathbf{9}_{(1,0,\kappa_+\kappa_-)}^0 \\ \mathbf{3}_{(-1,0,\kappa_-)}^- &\otimes \mathbf{3}_{(-1,0,\kappa_-)}^- &= \mathbf{9}_{(1,0,\kappa_-^2)}^- \end{aligned} \quad (5.15)$$

where right hand sides of eqs(5.14-5.15) are given by

$$\begin{aligned} \mathbf{25}_{(1,1,0)}^0 &= \mathbf{20}_{(0,-1,0)}^0 \oplus \mathbf{4}_{(0,1,-1)}^0 \oplus \mathbf{1}_{(1,1,1)}^0 \\ \mathbf{20}_{(0,-1,0)}^0 &= \mathbf{15}_{(-1,0,0)}^0 \oplus \mathbf{5}_{(1,-1,0)}^0 \\ \mathbf{16}_{(0,1,1)}^0 &= \mathbf{15}_{(-1,0,0)}^0 \oplus \mathbf{1}_{(1,1,1)}^0 \\ \mathbf{15}_{(-1,0,0)}^0 &= \mathbf{9}_{(1,0,-1)}^0 \oplus \mathbf{6}_{(-2,0,1)}^0 \\ \mathbf{6}_{(-2,0,1)}^0 &= \mathbf{3}_{(-1,0,\kappa_+)}^+ \oplus \mathbf{3}_{(-1,0,\kappa_-)}^- \end{aligned} \quad (5.16)$$

and

$$\begin{aligned} \mathbf{12}_{(0,0,-\kappa_+)}^+ &= \mathbf{9}_{(1,0,-1)}^0 \oplus \mathbf{3}_{(-1,0,\kappa_-)}^- \\ \mathbf{12}_{(0,0,-\kappa_-)}^- &= \mathbf{9}_{(1,0,-1)}^0 \oplus \mathbf{3}_{(-1,0,\kappa_+)}^+ \\ \mathbf{9}_{(1,0,-1)}^0 &= \mathbf{5}_{(1,-1,0)}^0 \oplus \mathbf{4}_{(0,1,-1)}^0 \\ \mathbf{9}_{(1,0,\kappa_+^2)}^+ &= \mathbf{5}_{(1,-1,0)}^0 \oplus \mathbf{3}_{(-1,0,\kappa_+)}^+ \oplus \mathbf{1}_{(1,1,1)}^0 \\ \mathbf{9}_{(1,0,\kappa_-^2)}^- &= \mathbf{5}_{(1,-1,0)}^0 \oplus \mathbf{3}_{(-1,0,\kappa_-)}^- \oplus \mathbf{1}_{(1,1,1)}^0 \end{aligned} \quad (5.17)$$

## 6 Conclusion

F-theory GUT models have a finite spectrum of localised matter curves  $\{\Phi_{R_i}\}$  indexed, in addition to gauge charges, by quantum numbers of monodromy  $\Gamma$ . In the example of

$SO_{10} \times \Gamma$  models, the possible  $\Gamma$ 's are given by subgroups of the symmetric  $\mathbb{S}_4$  as depicted on figure 1. In the particular  $SO_{10} \times \mathbb{S}_4$  model, localised matter curves is as in table (3.5) involving, in addition to 16-plets and 10-plets, flavons  $\vartheta$  carrying non trivial charges under monodromy  $\mathbb{S}_4$ . The same properties hold for  $SO_{10} \times \Gamma$  models with monodromy  $\Gamma \subset \mathbb{S}_4$ ; the main difference is that now the numbers  $n_i^\Gamma$  and the dimensions  $d_i^\Gamma$  of irreducible representations  $\mathbf{R}_i^\Gamma$  of monodromy group  $\Gamma$  are smaller as shown on eqs(4.1,4.6,4.11); but the corresponding matter spectrums  $\{\Phi_{R_i}\}$  still carry charges under  $\Gamma$  including the flavons which play an important role in models building; in particular in the study of neutrino mixing and the Higgs sector of extended MSSM as well as for dealing with GUT constraints such as proton decay. By requiring invariance under monodromy  $\Gamma$ , one disposes therefore of an important tool to deal with constructing general superpotentials involving flavons. This construction, requires however the fusion rules (1.2). The same think can be said about  $SU_5 \times \mathbb{S}_5$  prototype and in general about the  $SU_5 \times \Gamma$  models where monodromies  $\Gamma$  are given by subgroups of  $\mathbb{S}_5$ .

In this work we have constructed the closed fusion algebras  $\mathcal{F}_\Gamma$  of the F-GUT operators spectrum  $\{\Phi_{R_i}\}$  indexed by  $\mathbf{R}_i^\Gamma$  representations of monodromy groups  $\Gamma$ ; in particular monodromy given by the symmetric groups  $\mathbb{S}_5$ ,  $\mathbb{S}_4$ ,  $\mathbb{S}_3$ ; the non abelian alternating  $\mathbb{A}_5$ ,  $\mathbb{A}_4$ ; and the dihedral  $\mathbb{D}_4$ . These  $\mathcal{F}_\Gamma$ 's are important for building monodromy invariant superpotentials  $W(\Phi_{R_i})$  for GUT models with gauge symmetry  $G = SO_{10}$  and  $SU_5$ . In the example of  $SO_{10} \times \Gamma$  theory, typical mass terms, generated by restricting to VEVs of Higgs and flavons, have the form  $16^i M_{ij} 16^j$  with mass matrix controlled by  $\mathcal{F}_\Gamma$  fusion relations.

To derive the  $\mathcal{F}_\Gamma$ 's structures obtained in this paper, we have used properties of the characters of the irreducible representations of discrete symmetries. Our construction, which may be used for other purposes, extends straightforwardly to any finite symmetry group; including products like  $\Gamma_1 \times \Gamma_2$ . We end this study by describing briefly the case of abelian symmetries  $\mathbf{H}$ ; they have completely reducible representations; and so simpler fusion algebras  $\mathcal{F}_\mathbf{H}$ . In the example of  $\mathbb{Z}_2$ , there are two irreducible representations  $\mathbf{1}_\pm$ ; and the corresponding fusion algebra  $\mathcal{F}_{\mathbb{Z}_2}$  is just  $\mathbf{1}_+ \otimes \mathbf{1}_+ = \mathbf{1}_+$ ,  $\mathbf{1}_- \otimes \mathbf{1}_- = \mathbf{1}_+$  and  $\mathbf{1}_- \otimes \mathbf{1}_+ = \mathbf{1}_-$ . For the case  $\mathbb{Z}_3$ , we have three irreducible 1-dim representations following from the  $\mathbb{Z}_3$  group property

$$3 = 1^2 + (1')^2 + (1'')^2$$

with characters given by the three cubic roots  $j^p$  of unity;  $j^3 = 1$ . By denoting these representations as  $\mathbf{1}_{j^p}$  with  $p = 0, 1, 2 \bmod 3$ , the corresponding fusion algebra  $\mathcal{F}_{\mathbb{Z}_3}$  is nothing but  $\mathbf{1}_{j^p} \otimes \mathbf{1}_{j^q} = \mathbf{1}_{j^{p+q}}$ . Extension to  $\mathcal{F}_{\mathbb{Z}_N}$  is straightforward.

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## 7 Appendix

In this appendix, we collect the character tables of discrete  $\Gamma$ 's appearing in F-theory GUT. The  $\mathfrak{C}_i$ 's refer to conjugacy classes; the  $\mathbf{R}_i$ 's for irreducible representations, and  $\chi_{\mathbf{R}_i}$ 's to characters.

- *permutation symmetry  $\mathbb{S}_4$*

$\mathfrak{C}_i \backslash \text{irrep } \mathbf{R}_j$	$\chi_I$	$\chi_{\mathbf{3}'}$	$\chi_2$	$\chi_3$	$\chi_\epsilon$	order
$\mathfrak{C}_1 \equiv e$	1	3	2	3	1	1
$\mathfrak{C}_2 \equiv (12)$	1	-1	0	1	-1	6
$\mathfrak{C}_3 \equiv (12)(34)$	1	-1	2	-1	1	3
$\mathfrak{C}_4 \equiv (123)$	1	0	-1	0	1	8
$\mathfrak{C}_5 \equiv (1234)$	1	1	0	-1	-1	6

(7.1)

- *permutation symmetry  $\mathbb{S}_3$*

$\mathfrak{C}_i \backslash \text{irrep } \mathbf{R}_j$	$\chi_I$	$\chi_2$	$\chi_\epsilon$	order
$\mathfrak{C}_1 \equiv e$	1	2	1	1
$\mathfrak{C}_2 \equiv (12)$	1	0	-1	3
$\mathfrak{C}_3 \equiv (123)$	1	-1	1	2

(7.2)

- *alternating group  $\mathbb{A}_4$*

$\mathfrak{C}_i \backslash \text{irrep } \mathbf{R}_j$	$\chi_I$	$\chi_{1'}$	$\chi_{1''}$	$\chi_3$	order
$\mathfrak{C}_1 \equiv e$	1	1	1	3	1
$\mathfrak{C}_2 \equiv (12)(34)$	1	1	1	-1	3
$\mathfrak{C}_3 \equiv (123)$	1	$j$	$j^2$	0	4
$\mathfrak{C}_4 \equiv (132)$	1	$j^2$	$j$	0	4

(7.3)

- *dihedral symmetry  $\mathbb{D}_4$*

$\mathfrak{C}_i \backslash \chi_{\mathbf{R}_j}$	$\chi_{1_1}$	$\chi_{1_2}$	$\chi_{1_3}$	$\chi_{1_4}$	$\chi_2$	order
$\mathfrak{C}_1$	1	1	1	1	2	1
$\mathfrak{C}_2$	1	1	1	1	-2	1
$\mathfrak{C}_3$	1	1	-1	-1	0	2
$\mathfrak{C}_4$	1	-1	1	-1	0	2
$\mathfrak{C}_5$	1	-1	-1	1	-0	2

(7.4)

- permutation symmetry  $S_5$

$\mathfrak{C}_i \backslash \chi_{R_j}$	$\chi_1$	$\chi_{1'}$	$\chi_4$	$\chi_{4'}$	$\chi_5$	$\chi_{5'}$	$\chi_6$	order
$\mathfrak{C}_1 = e$	1	1	4	4	5	5	6	1
$\mathfrak{C}_2 = (12)$	1	-1	2	-2	1	-1	0	10
$\mathfrak{C}_3 = (12)(34)$	1	1	0	0	1	1	-2	15
$\mathfrak{C}_4 = (123)$	1	1	1	1	-1	-1	0	20
$\mathfrak{C}_5 = (1234)$	1	-1	0	0	-1	1	0	30
$\mathfrak{C}_6 = (123)(45)$	1	-1	-1	1	1	-1	0	20
$\mathfrak{C}_7 = (12345)$	1	1	-1	-1	0	0	1	24

(7.5)

- alternating group  $A_5$

$\mathfrak{C}_i \backslash \chi_{R_j}$	$\chi_1$	$\chi_3$	$\chi_{3'}$	$\chi_4$	$\chi_5$	order
$\mathfrak{C}_1 = e$	1	3	3	4	5	1
$\mathfrak{C}_2 = (12)(34)$	1	-1	-1	0	1	15
$\mathfrak{C}_3 = (123)$	1	0	0	1	-1	20
$\mathfrak{C}_4 = (12345)$	1	$\kappa_+$	$\kappa_-$	-1	0	12
$\mathfrak{C}_5 = (13524)$	1	$\kappa_-$	$\kappa_+$	-1	0	12

(7.6)

with  $\kappa_{\pm} = \frac{1 \pm \sqrt{5}}{2}$ ,  $\kappa_+ + \kappa_- = 1$ ,  $\kappa_- \kappa_+ = -1$ ,  $\kappa_+^2 = 1 + \kappa_+$  and  $\kappa_-^2 = 1 + \kappa_- = 2 - \kappa_+$ .

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